

## ON THE EXISTENCE OF RAMIFIED ABELIAN COVERS

VALERY ALEXEEV AND RITA PARDINI

ABSTRACT. Given a normal complete variety  $Y$ , distinct effective Weil divisors  $D_1, \dots, D_n$  of  $Y$  and positive integers  $d_1, \dots, d_n$ , we spell out the conditions for the existence of an abelian cover  $X \rightarrow Y$  branched with order  $d_i$  on  $D_i$  for  $i = 1, \dots, n$ .

As an application, we prove that a Galois cover of a normal complete toric variety branched on the torus-invariant divisors is itself a toric variety.

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*Dedicated to Alberto Conte on his 70<sup>th</sup> birthday.*

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## 1. Introduction

Given a projective variety  $Y$  and effective divisors  $D_1, \dots, D_n$  of  $Y$ , deciding whether there exists a Galois cover branched on  $D_1, \dots, D_n$  with given multiplicities is a very complicated question, which in the complex case is essentially equivalent to describing the finite quotients of the fundamental group of  $Y \setminus (D_1 \cup \dots \cup D_n)$ .

In Section 2 of this paper we answer this question for a normal variety  $Y$  in the case that the Galois group of the cover is abelian (Theorem 2.1), using the theory developed in [Par91] and [AP12]. In particular, we prove that when the class group  $\text{Cl}(Y)$  is torsion free, every abelian cover of  $Y$  branched on  $D_1, \dots, D_n$  with given multiplicities is the quotient of a maximal such cover, unique up to isomorphism.

In Section 3 we analyze the same question using toric geometry in the case when  $Y$  is a normal complete toric variety and  $D_1, \dots, D_n$  are invariant divisors and obtain results that parallel those in Section 2 (Theorem 3.5). Combining the two approaches we are able to show that a Galois cover of a normal complete toric variety branched on the invariant divisors is toric (Theorem 3.6).

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**Notation.**  $G$  always denotes a finite group, almost always abelian, and  $G^* := \text{Hom}(G, \mathbb{K})^*$  the group of characters;  $o(g)$  is the order of the element  $g \in G$  and  $|H|$  is the cardinality of a subgroup  $H < G$ . We work over an algebraically closed field  $\mathbb{K}$  whose characteristic does not divide the order of the finite abelian groups

we consider.

If  $A$  is an abelian group we write  $A[d] := \{a \in A \mid da = 0\}$  ( $d$  an integer),  $A^\vee := \text{Hom}(A, \mathbb{Z})$  and we denote by  $\text{Tors}(A)$  the torsion subgroup of  $A$ .

The smooth part of a variety  $Y$  is denoted by  $Y_{\text{sm}}$ . The symbol  $\equiv$  denotes linear equivalence of divisors. If  $Y$  is a normal variety we denote by  $\text{Cl}(Y)$  the group of classes, namely the group of Weil divisors up to linear equivalence.

## 2. Abelian covers

**2.1. The fundamental relations.** We quickly recall the theory of abelian covers (cf. [Par91], [AP12], and also [PT95]) in the most suitable form for the applications considered here.

There are slightly different definitions of abelian covers in the literature (see, for instance, [AP12] that treats also the non-normal case). Here we restrict our attention to the case of normal varieties, but we do not require that the covering map be flat, hence we define an abelian cover as a finite Galois morphism  $\pi: X \rightarrow Y$  of normal varieties with abelian Galois group  $G$  ( $\pi$  is also called a “ $G$ -cover”).

Recall that, as already stated in the Notations, throughout all the paper we assume that  $G$  has order not divisible by  $\text{char } \mathbb{K}$ .

To every component  $D$  of the branch locus of  $\pi$  we associate the pair  $(H, \psi)$ , where  $H < G$  is the cyclic subgroup consisting of the elements of  $G$  that fix the preimage of  $D$  pointwise (the *inertia subgroup* of  $D$ ) and  $\psi$  is the element of the character group  $H^*$  given by the natural representation of  $H$  on the normal space to the preimage of  $D$  at a general point (these definitions are well posed since  $G$  is abelian). It can be shown that  $\psi$  generates the group  $H^*$ .

If we fix a  $d$ -th root  $\zeta$  of 1, where  $d$  is the exponent of the group  $G$ , then a pair  $(H, \psi)$  as above is determined by the generator  $g \in H$  such that  $\psi(g) = \zeta^{\frac{d}{o(g)}}$ . We follow this convention and attach to every component  $D_i$  of the branch locus of  $\pi$  a nonzero element  $g_i \in G$ .

If  $\pi$  is flat, which is always the case when  $Y$  is smooth, the sheaf  $\pi_* \mathcal{O}_X$  decomposes under the  $G$ -action as  $\bigoplus_{\chi \in G^*} L_\chi^{-1}$ , where the  $L_\chi$  are line bundles ( $L_1 = \mathcal{O}_Y$ ) and  $G$  acts on  $L_\chi^{-1}$  via the character  $\chi$ .

Given  $\chi \in G^*$  and  $g \in G$ , we denote by  $\overline{\chi}(g)$  the smallest non-negative integer  $a$  such that  $\chi(g) = \zeta^{\frac{ad}{o(g)}}$ . The main result of [Par91] is that the  $L_\chi$ ,  $D_i$  (the *building data* of  $\pi$ ) satisfy the following *fundamental relations*:

$$(2.1) \quad L_\chi + L_{\chi'} \equiv L_{\chi+\chi'} + \sum_i \varepsilon_{\chi, \chi'}^i D_i \quad \forall \chi, \chi' \in G^*$$

where  $\varepsilon_{\chi, \chi'}^i = \lfloor \frac{\overline{\chi}(g_i) + \overline{\chi'}(g_i)}{o(g_i)} \rfloor$ . (Notice that the coefficients  $\varepsilon_{\chi, \chi'}^i$  are equal either to 0 or to 1). Conversely, distinct irreducible divisors  $D_i$  and line bundles  $L_\chi$  satisfying (2.1) are the building data of a flat (normal)  $G$ -cover  $X \rightarrow Y$ ; in addition, if  $h^0(\mathcal{O}_Y) = 1$  then  $X \rightarrow Y$  is uniquely determined up to isomorphism of  $G$ -covers.

If we fix characters  $\chi_1, \dots, \chi_r \in G^*$  such that  $G^*$  is the direct sum of the subgroups generated by the  $\chi_j$ , and we set  $L_j := L_{\chi_j}$ ,  $m_j := o(\chi_j)$ , then the solutions of the fundamental relations (2.1) are in one-one correspondence with the solutions of the

following *reduced fundamental relations*:

$$(2.2) \quad m_j L_j \equiv \sum_i \frac{m_j \overline{\chi_j}(g_i)}{d_i} D_i, \quad i = 1, \dots, r$$

Notice that if  $\text{Pic}(Y)[d] = 0$ , then for fixed  $(D_i, g_i)$ ,  $i = 1, \dots, n$ , the solution of (2.2) is unique, hence the *branch data*  $(D_i, g_i)$  determine the cover.

In order to deal with the case when  $Y$  is normal but not smooth, we observe that the complement  $Y \setminus Y_{\text{sm}}$  of the smooth part has codimension  $> 1$ ; so any cover  $X \rightarrow Y$  can be recovered from its restriction  $X' \rightarrow Y_{\text{sm}}$  to the smooth locus, by taking the integral closure of  $Y$  in the extension in  $\mathbb{K}(X') \supset \mathbb{K}(Y)$ . In addition, if  $h^0(\mathcal{O}_Y) = 1$ , then we have also  $h^0(\mathcal{O}_{Y_{\text{sm}}}) = 1$  and the cover  $X' \rightarrow Y_{\text{sm}}$  is determined by the building data  $L_\chi, D_i$ . Using the identification  $\text{Pic}(Y_{\text{sm}}) = \text{Cl}(Y_{\text{sm}}) = \text{Cl}(Y)$ , we can regard the  $L_\chi$  as elements of  $\text{Cl}(Y)$  and, taking the closure, the  $D_i$  as Weil divisors on  $Y$ , and we can interpret the fundamental relations as equalities in  $\text{Cl}(Y)$ . In this sense, if  $Y$  is normal variety with  $h^0(\mathcal{O}_Y) = 1$ , then the  $G$ -covers  $X \rightarrow Y$  are determined by the building data up to isomorphism.

**2.2. The maximal cover.** Let  $Y$  be a complete normal variety, let  $D_1, \dots, D_n$  be irreducible effective divisors of  $Y$  and let  $d_1, \dots, d_n$  be positive integers (it is convenient to allow the possibility that  $d_i = 1$  for some  $i$ ). We set  $d := \text{lcm}(d_1, \dots, d_n)$ .

We say that an abelian cover  $\pi: X \rightarrow Y$  is *branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$*  if:

- $\pi$  is totally ramified, i.e. the inertia subgroups of the divisorial components of the branch locus of  $\pi$  generate  $G$  (equivalently,  $\pi$  does not factorize through a cover  $X' \rightarrow Y$  that is étale over  $Y_{\text{sm}}$ );
- the divisorial part of the branch locus of  $\pi$  is contained in  $\sum_i D_i$ ;
- the ramification order of  $\pi$  over  $D_i$  is equal to  $d_i$ .

Let  $\eta: \tilde{Y} \rightarrow Y$  be a resolution of the singularities and set  $N(Y) := \text{Cl}(Y)/\eta_* \text{Pic}^0(\tilde{Y})$ . Since the map  $\eta_*: \text{Pic}(\tilde{Y}) = \text{Cl}(\tilde{Y}) \rightarrow \text{Cl}(Y)$  is surjective,  $N(Y)$  is a quotient of the Néron-Severi group  $\text{NS}(\tilde{Y})$ , hence it is finitely generated. It follows that  $\eta_* \text{Pic}^0(\tilde{Y})$  is the largest divisible subgroup of  $\text{Cl}(Y)$  and therefore  $N(Y)$  does not depend on the choice of the resolution of  $Y$  (this is easily checked also by a geometrical argument). The group  $\text{Cl}(Y)^\vee$  coincides with  $N(Y)^\vee$ , hence it is a finitely generated free abelian group of rank equal to the rank of  $N(Y)$ .

Consider the map  $\mathbb{Z}^n \rightarrow \text{Cl}(Y)$  that maps the  $i$ -th canonical generator to the class of  $D_i$ , let  $\phi: \text{Cl}(Y)^\vee \rightarrow \oplus_{i=1}^n \mathbb{Z}_{d_i}$  be the map obtained by composing the dual map  $\text{Cl}(Y)^\vee \rightarrow \oplus (\mathbb{Z}^n)^\vee$  with  $(\mathbb{Z}^n)^\vee = \mathbb{Z}^n \rightarrow \oplus_{i=1}^n \mathbb{Z}_{d_i}$  and let  $K_{\min}$  be the image of  $\phi$ . Let  $G_{\max}$  be the abelian group defined by the exact sequence:

$$(2.3) \quad 0 \rightarrow K_{\min} \rightarrow \oplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G_{\max} \rightarrow 0.$$

Then we have the following:

**Theorem 2.1.** *Let  $Y$  be a normal variety with  $h^0(\mathcal{O}_Y) = 1$ , let  $D_1, \dots, D_n$  be distinct irreducible effective divisors, let  $d_1, \dots, d_n$  be positive integers and set  $d := \text{lcm}(d_1, \dots, d_n)$ . Then:*

- (1) *If  $X \rightarrow Y$  is a  $G$ -cover branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$ , then:*
  - (a) *the map  $\oplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$  that maps  $1 \in \mathbb{Z}_{d_i}$  to  $g_i$  descends to a surjection  $G_{\max} \twoheadrightarrow G$ ;*

- (b) the map  $\mathbb{Z}_{d_i} \rightarrow G_{\max}$  is injective for every  $i = 1, \dots, n$ .
- (2) If the map  $\mathbb{Z}_{d_i} \rightarrow G_{\max}$  is injective for  $i = 1, \dots, n$  and  $N(Y)[d] = 0$ , then there exists a maximal abelian cover  $X_{\max} \rightarrow Y$  with Galois group  $G_{\max}$  branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$ .
- (3) If the map  $\mathbb{Z}_{d_i} \rightarrow G_{\max}$  is injective for  $i = 1, \dots, n$  and  $\text{Cl}(Y)[d] = 0$ , then the cover  $X_{\max} \rightarrow Y$  is unique up to isomorphism of  $G_{\max}$ -covers and every abelian cover  $X \rightarrow Y$  branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$  is a quotient of  $X_{\max}$  by a subgroup of  $G_{\max}$ .

*Proof.* Let  $H_1, \dots, H_t \in N(Y)$  be elements whose classes are free generators of the abelian group  $N(Y)/\text{Tors}(N(Y))$ , and write:

$$(2.4) \quad D_i = \sum_{j=1}^t a_{ij} H_j \pmod{\text{Tors}(N(Y))}, \quad j = 1, \dots, t$$

Hence, the subgroup  $K_{\min}$  of  $\oplus_{i=1}^n \mathbb{Z}_{d_i}$  is generated by the elements  $z_j := (a_{1j}, \dots, a_{nj})$ , for  $j = 1, \dots, t$ .

Let  $X \rightarrow Y$  be a  $G$ -cover branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$  and let  $(D_i, g_i)$  be its branch data. Consider the map  $\oplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$  that maps  $1 \in \mathbb{Z}_{d_i}$  to  $g_i$ ; this map is surjective, by the assumption that  $X \rightarrow Y$  is totally ramified, and its restriction to  $\mathbb{Z}_{d_i}$  is injective for  $i = 1, \dots, n$ , since the cover is branched on  $D_i$  with order  $d_i$ . If we denote by  $K$  the kernel of  $\oplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$ , to prove (1) it suffices to show that  $K \supseteq K_{\min}$ . Dually, this is equivalent to showing that  $G^* \subseteq K_{\min}^\perp \subset \oplus_{i=1}^n (\mathbb{Z}_{d_i})^*$ . Let  $\psi_i \in (\mathbb{Z}_{d_i})^*$  be the generator that maps  $1 \in \mathbb{Z}_{d_i}$  to  $\zeta^{\frac{d}{d_i}}$  and write  $\chi \in G^*$  as  $(\psi_1^{b_1}, \dots, \psi_n^{b_n})$ , with  $0 \leq b_i < d_i$ ; if  $o(\chi) = m$  then (2.2) gives  $mL_\chi \equiv \sum_{i=1}^n \frac{mb_i}{d_i} D_i$ . Plugging (2.4) in this equation we obtain that  $\sum_{i=1}^n \frac{b_i a_{ij}}{d_i}$  is an integer for  $j = 1, \dots, t$ , namely  $\chi \in K_{\min}^\perp$ .

(2) Let  $\chi_1, \dots, \chi_r$  be a basis of  $G^*$  and, as above, for  $s = 1, \dots, r$  write  $\chi_s = (\psi_1^{b_{s1}}, \dots, \psi_n^{b_{sn}})$ , with  $0 \leq b_{si} < d_i$ . Since by assumption  $N(Y)[d] = 0$ , by the proof of (1) the elements  $\sum_{j=1}^t (\sum_{i=1}^n \frac{b_{si} a_{ij}}{d_i}) H_j$ ,  $s = 1, \dots, r$ , can be lifted to solutions  $\overline{L}_s \in N(Y)$  of the reduced fundamental relations (2.2) for a  $G_{\max}$ -cover with branch data  $(D_i, g_i)$ , where  $g_i \in G$  is the image of  $1 \in \mathbb{Z}_{d_i}$ . Since the group  $\text{Pic}^0(Y)$  is divisible, it is possible to lift the  $\overline{L}_s$  to solutions  $L_s \in \text{Cl}(Y)$ . We let  $X_{\max} \rightarrow Y$  be the  $G_{\max}$ -cover determined by these solutions.

(3) Since  $\text{Cl}(Y)[d] = 0$ , any  $G$ -cover such that the exponent of  $G$  is a divisor of  $d$  is determined uniquely by the branch data; in particular, this holds for the cover  $X_{\max} \rightarrow Y$  in (2) and for every intermediate cover  $X_{\max}/H \rightarrow Y$ , where  $H < G_{\max}$ . The claim now follows by (1).  $\square$

**Example 2.1.** Take  $Y = \mathbb{P}^{n-1}$  and let  $D_1, \dots, D_n$  be the coordinate hyperplanes. In this case the group  $K$  is generated by  $(1, \dots, 1) \in \oplus_{i=1}^n \mathbb{Z}_{d_i}$ , hence by Theorem 2.1 there exists an abelian cover of  $\mathbb{P}^{n-1}$  branched over  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$  iff  $d_i$  divides  $\text{lcm}(d_1, \dots, \widehat{d_i}, \dots, d_n)$  for every  $i = 1, \dots, n$ . For  $d_1 = \dots = d_n = d$ , then  $G_{\max} = \mathbb{Z}_d^n / \langle (1, \dots, 1) \rangle$  and  $X_{\max} \rightarrow \mathbb{P}^{n-1}$  is the cover  $\mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$  defined by  $[x_1, \dots, x_n] \mapsto [x_1^d, \dots, x_n^d]$ .

In general,  $X_{\max}$  is a weighted projective space  $\mathbb{P}(\frac{d}{d_1}, \dots, \frac{d}{d_n})$  and the cover is given by  $[x_1, \dots, x_n] \mapsto [x_1^{\frac{d}{d_1}}, \dots, x_n^{\frac{d}{d_n}}]$ .

### 3. Toric covers

**Notations 3.1.** Here, we fix the notations which are standard in toric geometry. A (complete normal) toric variety  $Y$  corresponds to a fan  $\Sigma$  living in the vector space  $N \otimes \mathbb{R}$ , where  $N \cong \mathbb{Z}^r$ . The dual lattice is  $M = N^\vee$ . The torus is  $T = N \otimes \mathbb{C}^* = \text{Hom}(M, \mathbb{C}^*)$ .

The integral vectors  $r_i \in N$  will denote the integral generators of the rays  $\sigma_i \in \Sigma(1)$  of the fan  $\Sigma$ . They are in a bijection with the  $T$ -invariant Weil divisors  $D_i$  ( $i = 1, \dots, n$ ) on  $Y$ .

**Definition 3.2.** A *toric cover*  $f: X \rightarrow Y$  is a finite morphism of toric varieties corresponding to the map of fans  $F: (N', \Sigma') \rightarrow (N, \Sigma)$  such that:

- (1)  $N' \subseteq N$  is a sublattice of finite index, so that  $N' \otimes \mathbb{R} = N \otimes \mathbb{R}$ .
- (2)  $\Sigma' = \Sigma$ .

The proof of the following lemma is immediate.

**Lemma 3.3.** *The morphism  $f$  has the following properties:*

- (1) *It is equivariant with respect to the homomorphism of tori  $T' \rightarrow T$ .*
- (2) *It is an abelian cover with Galois group  $G = \ker(T' \rightarrow T) = N/N'$ .*
- (3) *It is ramified only along the boundary divisors  $D_i$ , with multiplicities  $d_i \geq 1$  defined by the condition that the integral generator of  $N' \cap \mathbb{R}_{\geq 0} r_i$  is  $d_i r_i$ .*

**Proposition 3.4.** *Let  $Y$  be a complete toric variety such that  $\text{Cl}(Y)$  is torsion free, and  $X \rightarrow Y$  be a toric cover. Then, with notations as above, there exists the following commutative diagram with exact rows and columns.*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Cl}(Y)^\vee & \longrightarrow & K & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} d_i D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} d_i \longrightarrow 0 \\
 & & \downarrow p' & & \downarrow p & & \downarrow \\
 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & G \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

(Here  $D_i^*$  are formal symbols denoting a basis of  $\mathbb{Z}^n$ , resp.  $\oplus_{i=1}^n \mathbb{Z} d_i$ ). Moreover, each of the homomorphisms  $\mathbb{Z} d_i \rightarrow G$  is an embedding.

*Proof.* The third row appeared in Lemma 3.3, and the second row is the obvious one.

It is well known that the boundary divisors on a complete normal toric variety span the group  $\text{Cl}(Y)$ , and that there exists the following short exact sequence of lattices:

$$0 \rightarrow M \rightarrow \oplus_{i=1}^n \mathbb{Z} D_i \rightarrow \text{Cl}(Y) \rightarrow 0.$$

Since  $\text{Cl}(Y)$  is torsion free by assumption, this sequence is split and dualizing it one obtains the central column. Since  $\bigoplus_{i=1}^n \mathbb{Z}D_i^* \rightarrow N$  is surjective, then so is  $\bigoplus_{i=1}^n \mathbb{Z}d_i \rightarrow G$ . The group  $K$  is defined as the kernel of this map.

Finally, the condition that  $\mathbb{Z}d_i \rightarrow G$  is injective is equivalent to the condition that the integral generator of  $N' \cap \mathbb{R}_{\geq 0}r_i$  is  $d_i r_i$ , which holds by Lemma 3.3.  $\square$

**Theorem 3.5.** *Let  $Y$  be a complete toric variety such that  $\text{Cl}(Y)$  is torsion free, let  $d_1, \dots, d_n$  be positive integers and let  $K_{\min}$  and  $G_{\max}$  be defined as in sequence (2.3). Then:*

- (1) *There exists a toric cover branched on  $D_i$  of order  $d_i$ ,  $i = 1, \dots, n$ , iff the map  $\mathbb{Z}d_i \rightarrow G_{\max}$  is injective for  $i = 1, \dots, n$ .*
- (2) *If condition (1) is satisfied, then among all the toric covers of  $Y$  ramified over the divisors  $D_i$  with multiplicities  $d_i$  there exists a maximal one  $X_{\text{Tmax}} \rightarrow Y$ , with Galois group  $G_{\max}$ , such that any other toric cover  $X \rightarrow Y$  with the same branching orders is a quotient  $X = X_{\text{Tmax}}/H$  by a subgroup  $H < G_{\max}$ .*

*Proof.* Let  $X \rightarrow Y$  be a toric cover branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$ , let  $N'$  be the corresponding sublattice of  $N$  and  $G = N/N'$  the Galois group. Let  $N'_{\min}$  be the subgroup of  $N$  generated by  $d_i r_i$ ,  $i = 1, \dots, n$ . Then one must have  $N'_{\min} \subseteq N'$ , hence the map  $\mathbb{Z}d_i \rightarrow N/N'_{\min}$  is injective since  $\mathbb{Z}d_i \rightarrow G = N/N'$  is injective by Proposition 3.4. We set  $X_{\text{Tmax}} \rightarrow Y$  to be the cover for  $N'_{\min}$ . Clearly, the cover for the lattice  $N'$  is a quotient of the cover for the lattice  $N'_{\min}$  by the group  $H = N'/N'_{\min}$ .

Consider the second and third rows of the diagram of Proposition 3.4 as a short exact sequence of 2-step complexes  $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ . The associated long exact sequence of cohomologies gives

$$\text{Cl}(Y)^\vee \longrightarrow K \longrightarrow \text{coker}(p') \longrightarrow 0$$

For  $N' = N'_{\min}$ , the map  $p'$  is surjective, hence  $\text{Cl}(Y)^\vee \rightarrow K$  is surjective too, and  $K = K_{\min}$ ,  $N/N'_{\min} = G_{\max}$ .

Vice versa, suppose that in the following commutative diagram with exact row and columns each of the maps  $\mathbb{Z}d_i \rightarrow G$  is injective.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Cl}(Y)^\vee & \xrightarrow{q} & K_{\min} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}d_i D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}d_i \longrightarrow 0 \\
 & & \downarrow p & & \downarrow & & \\
 & & N & & G_{\max} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

We complete the first row on the left by adding  $\ker(q)$ . We have an induced homomorphism  $\ker(q) \rightarrow \bigoplus \mathbb{Z}d_i D_i^*$ , and we define  $N'$  to be its cokernel.

Now consider the completed first and second rows as a short exact sequence of 2-step complexes  $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ . The associated long exact sequence of cohomologies says that  $\ker(q) \rightarrow \oplus_{i=1}^n \mathbb{Z}d_i D_i^*$  is injective, and the sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow G_{\max} \longrightarrow 0$$

is exact. It follows that  $N' = N'_{\min}$  and the toric morphism  $(N'_{\min}, \Sigma) \rightarrow (N, \Sigma)$  is then the searched-for maximal abelian toric cover.  $\square$

We now combine the results of this section with those of §2 to obtain a structure result for Galois covers of toric varieties.

**Theorem 3.6.** *Let  $Y$  be a normal complete toric variety and let  $X \rightarrow Y$  be a Galois cover with group  $G$  branched on the invariant divisors  $D_1, \dots, D_n$ . Then  $G$  is abelian and  $X \rightarrow Y$  is a toric cover.*

*Proof.* Let  $U \subset Y$  be the open orbit and let  $X' \rightarrow U$  be the cover obtained by restriction. By [Mi, Prop. 1],  $X' \rightarrow U$  is, up to isomorphism, a homomorphism of tori (recall that  $|G|$  is not divisible by  $\text{char } \mathbb{K}$  by assumption). It follows that the Galois group  $G$  is abelian.

Let  $d_1, \dots, d_n$  be the orders of ramification of  $X \rightarrow Y$  on  $D_1, \dots, D_n$ . Assume first that  $\text{Cl}(Y)$  has no torsion. (Notice that in this case any abelian cover of  $Y$  is totally ramified). Then by Theorem 2.1 every abelian cover branched on  $D_1, \dots, D_n$  with orders  $d_1, \dots, d_n$  is a quotient of the maximal abelian cover  $X_{\max} \rightarrow Y$  by a subgroup  $H < G_{\max}$ . In particular, this is true for the cover  $X_{\text{Tmax}} \rightarrow Y$  of Theorem 3.5. Since  $X_{\max}$  and  $X_{\text{Tmax}}$  have the same Galois group it follows that  $X_{\max} = X_{\text{Tmax}}$ . Hence  $X \rightarrow Y$ , being a quotient of  $X_{\text{Tmax}}$ , is a toric cover.

Consider now the general case. The group  $\text{Tors Cl}(Y)$  is finite, and isomorphic to  $N/\langle r_i \rangle$ . The cover  $Y' \rightarrow Y$  corresponding to  $\text{Tors Cl}(Y)$  is toric, and one has  $\text{Tors Cl}(Y') = 0$ .

Indeed, on a toric variety the group  $\text{Cl}(Y)$  is generated by the  $T$ -invariant Weil divisors  $D_i$ . Thus,  $\text{Cl}(Y)$  is the quotient of the free abelian group  $\oplus \mathbb{Z}D_i$  of all  $T$ -invariant divisors modulo the subgroup  $M$  of principal  $T$ -invariant divisors. Thus,  $\text{Tors Cl}(Y) \simeq M'/M$ , where  $M \subset M' \subset \oplus \mathbb{Q}D_i$  is the subgroup of  $\mathbb{Q}$ -linear functions on  $N$  taking integral values on the vectors  $r_i$  defined in 3.1. Then  $N' := M'^\vee$  is the subgroup of  $N$  generated by the  $r_i$ , and the cover  $Y' \rightarrow Y$  is the cover corresponding to the map of fans  $(N', \Sigma) \rightarrow (N, \Sigma)$ . On  $Y'$  one has  $N' = \langle r_i \rangle$ , so  $\text{Tors Cl}(Y') = 0$ .

Let  $X' \rightarrow Y'$  be a connected component the pull back of  $X \rightarrow Y$ : it is an abelian cover branched on the invariant divisors of  $Y'$ , hence by the first part of the proof it is toric. The map  $X' \rightarrow Y$  is toric, since it is a composition of toric morphisms, hence the intermediate cover  $X \rightarrow Y$  is also toric.  $\square$

**Remark 3.7.** The argument that shows that the Galois group is abelian in the proof of Theorem 3.6 was suggested to us by Angelo Vistoli. He also remarked that it is possible to prove Theorem 3.6 in a more conceptual way by showing that the torus action on the cover  $X' \rightarrow U$  of the open orbit of  $Y$  extends to  $X$ , in view of the properties of the integral closure. However our approach has the advantage of describing explicitly the fan/building data associated with the cover.

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